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On asymptotic modelling of constraints that do not change the degrees of freedom

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Abstract

This paper shows that the asymptotic modelling method using positive and negative restraints may be applied to systems where the constraints do not affect the number of degrees of freedom. This applies in cases where the constraints are not associated with mass or inertia. The existence and convergence theorems have been refined to cover this type of systems.

1. Introduction

Asymptotic modelling is a popular method [1–5], introduced by Courant [1] to remove the limitation on the choice of admissible functions in the Rayleigh–Ritz method, and later adopted in many disciplines including the finite elements [2]. The recent introduction of the use of negative restraints or penalty parameters [3–5] has made this procedure more reliable as it is possible to determine the maximum possible error due to the violation of a constraint due to the asymptotic modeling. A number of publications that use the method are listed in Refs. [3–5].

Two recently published theorems on the existence of natural frequencies of systems with artificial restraints and their convergence justify the use of positive and negative restraints in asymptotic modelling [4]. The theorems show that if h restraints of positive or negative stiffness are added to an n degree of freedom system (A) where $h<n$, then for the resulting system (A_h), there exist at least (n–h) natural frequencies and modes and that as the h stiffness parameters approach infinity, the (n–h) natural frequencies and modes of System A_h would asymptotically approach those of the n degree of freedom system subject to h constraints ($\tilde{A}_h$). In deriving these theorems the constraints were assumed to prevent the motion of a mass or an inertial element so as to reduce the number of degrees of freedom by the number of constraints. There are cases where a constraint may be applied in such a way that it does not alter the vibratory degrees of freedom of a system but can alter its shape. For example, consider the case of a massless cantilever beam carrying a particle of mass m_0 at its free end as shown in Fig. 1a. This is a single degree of freedom system. If a constraint in the form of a prop support were introduced as shown in Fig. 1b, its vibratory form would be constrained to have zero

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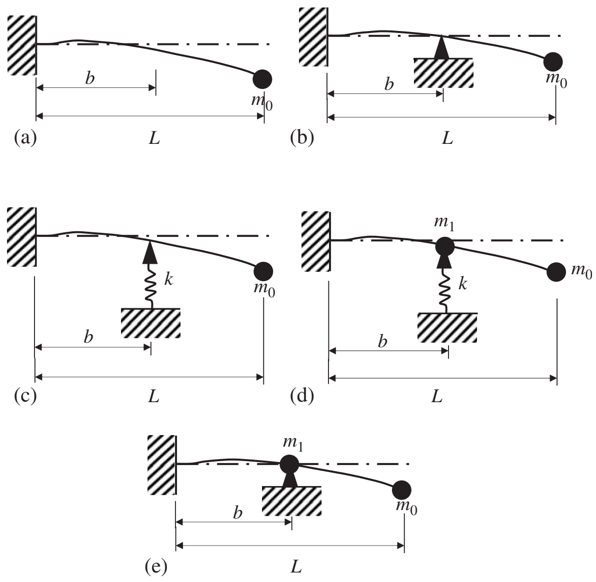

Fig. 1. Constrained, restrained and unrestrained systems: (a) System A, (b) System $\tilde{A}_{(1)}$, (c) System $A_{(1)}$, (d) System B_1, and (e) System $\tilde{B}_1$.

displacement at the prop but it would still retain its status of single degree of freedom system as the mass can vibrate. The purpose of this note is to show how the asymptotic modelling method using positive and negative restraints [4] may be applied to such systems where constraints do not affect the number of degrees of freedom. The existence and convergence theorems in Ref. [4] have been refined to cover the cases where the constraints are not associated with mass or inertia.

2. Derivations and results

The systems in Figs. 1a and b, being single degree of freedom systems, have one and only one natural frequency, which for each case may be determined exactly, or approximately, in the latter case using the Rayleigh–Ritz method. For convenience these systems are labeled A and $\tilde{A}_{(1)}$, respectively. The notation is consistent with that used in Ref. [4] but the subscript 1 is given in brackets to indicate that the constraint only changes the form of vibration and not the degrees of freedom. Taking the clamped end of the beam as the origin of the x-coordinate, if functions of the type

$$f(x) = \sum_{1}^{n} a_i (x/L)^{i+1} \tag{1}$$

were used for the lateral vibratory deflection of the beam in a Rayleigh–Ritz procedure, the resulting expression for the natural frequency would be an upperbound estimate of System A as these functions satisfy the admissibility requirement that the deflection of the beam and its slope are zero at the clamped end. For System A it is possible to obtain exact results with only two terms in the above series because the actual mode consists only of quadratic and cubic functions.

The potential energy of the beam is given by

$$V_m = \int_0^L (EI/2)(f'')^2 \, dx. \tag{2}$$

Here, EI and L are the flexural rigidity and length of the beam, respectively.

The kinetic energy is

$$T_m = \omega^2 \Psi_m \quad \text{where } \Psi_m = (m_0/2)(f(L))^2 \tag{3}$$

in which m_0 is the magnitude of the attached mass.

The Rayleigh–Ritz minimization equations $(\partial V_m/\partial a_i) - \omega^2(\partial \Psi_m/\partial a_i) = 0$ result in the following eigenvalue equation:

$$[K]\{a\} - \omega^2[M]\{a\} = \{0\}, \tag{4}$$

where

$$K_{i,j} = \frac{(i+1)i(j+1)jEI}{(i+j-1)L^3} \tag{5}$$

and

$$M_{i,j} = m_0. \tag{6}$$

The natural frequencies and modes of System A may be obtained by solving Eq. (4). The functions given by Eq. (1) are not admissible for System $\tilde{A}_{(1)}$. However, numerical results for the deflection of a propped cantilever obtained using the above series with positive and negative penalty functions—a generic term for the artificial restraints—to enforce constraints show that it is possible to determine a true lower bound estimate of the deflection in this way [5]. The proof presented in the present paper shows that these functions may be used for frequency calculations if the system is modelled asymptotically, and an upperbound estimate of the natural frequency obtained, by replacing the prop with a spring support with large positive and negative stiffness values as shown in Fig. 1c. This system is labelled $A_{(1)}$. The subscript 1 in bracket indicates that this system is obtained by applying a single restraint to System A along a coordinate that does not involve the motion of a mass or inertia. (In cases where a restraint is applied along a coordinate that involves the motion of mass or inertia the subscript would be given without brackets.)

The results for System $A_{(1)}$ may be generated by including the strain energy of the restraint in the total potential energy. Eq. (2) would change to

$$V_m = \int_0^L (EI/2)(f'')^2 \mathrm{d}x + (k/2)(f(b))^2, \tag{2a}$$

where b is the x-coordinate of the point at which the restraint is located and k is the stiffness of the restraint. This would give the following expression for the stiffness coefficient:

$$K_{i,j} = \frac{(i+1)i(j+1)jEI}{(i+j-1)L^3} + k\left(\frac{b}{L}\right)^{i+j+2}. \tag{5a}$$

The results for the natural frequency for $n = 6$ and $b = 0.5$ (the beam restrained at mid-span) are presented in Fig. 2. A frequency parameter μ given by $\mu = \omega^2 mL^3/EI$ is plotted against the stiffness of the artificial restraint defined non-dimensionally as $\eta = kL^3/EI$. Using the static stiffness values [6] tabulated for the cantilever case and determined by applying the exact slope-deflection equations for the propped cantilever, the natural frequency parameter of systems A and $(\tilde{A}_{(1)})$ were found to be 3.0 and 13.8, respectively. As can be seen the frequency parameter for the constrained system (dotted line in Fig. 2) is bounded by the values obtained using an artificial restraint with large positive and negative stiffness. Since this system has a discontinuity at $x = b$, the series in Eq. (1) does not give exact results but only upperbound values. Nevertheless, they agree well with exact results and the discrepancy cannot be noticed in the plot. As can be seen from this figure, the natural frequency asymptotically approaches the frequency of the constrained system as the magnitude of stiffness takes very large values, irrespective of the sign. The frequency becomes zero at a critical stiffness value, which in this case is about $-3\,EI/b^3$ ($\eta = -24$). There is a range in the negative stiffness values in which the frequency parameter is negative indicating the absence of a natural frequency. The curve has a pole at a stiffness value of approximately $-111\,EI/L^3$ and the system has no real natural frequency for $-111 < \eta < -24$. In Ref. [5], the pole occurs at a point where the overall stiffness of the structure is zero, corresponding in the present case to $\eta = -24$. However, in the frequency squared versus stiffness plot the pole is at the point where the structural stiffness changes from negative to positive infinity.

The theorems presented in [4] do not explain the convergence seen in this example. To understand this let us introduce an imaginary mass of finite but adjustable magnitude m_1 at the point where the restraint is applied resulting in System B_1 as shown in Fig. 1d. This changes the expression for the elements of the mass matrix

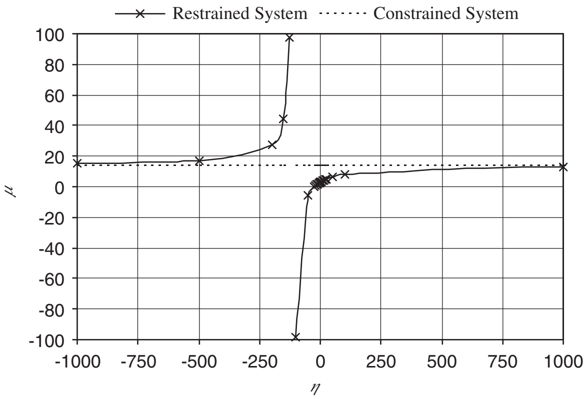

Fig. 2. Variation of the natural frequency with the stiffness of an artificial restraint.

given by Eq. (6) to

$$M_{i,j} = m_0 + m_1\left(\frac{b}{L}\right)^{i+j+2}. \tag{6a}$$

This system is a two degree of freedom system, and by the existence and convergence theorems [4] it will have at least one natural frequency, and as the magnitude of the stiffness parameter k approaches infinity one of its natural frequencies would asymptotically approach that of the corresponding constrained System $\tilde{B}_1$ (see Fig. 1e) as follows:

$$\text{As } k \to \infty, \quad \omega_{1,1} \to \tilde{\omega}_{1,1} \quad \text{and} \quad \omega_{2,1} \to \infty. \tag{7a,b}$$

For positive stiffness, the frequency of the constrained system is approached from below:

$$\text{As } k \to -\infty, \quad \omega_{2,1} \to \tilde{\omega}_{1,1} \quad \text{and} \quad \omega_{1,1} \notin \Re. \tag{8a,b}$$

In this case, the frequency of the constrained system is approached from above.

Eq. (8b) states that for very large negative k, the lowest frequency is imaginary. It should be noted here that the magnitude of the mass m_1 will have no effect on the frequency of the constrained system $\tilde{B}_1$ as the motion of the mass is prevented. Therefore, the natural frequency of Systems $\tilde{A}_{(1)}$ and $\tilde{B}_1$ must be equal. This means an asymptotic model B_1 may be used to obtain the natural frequency of $\tilde{A}_{(1)}$. Since the magnitude of m_1 has no influence on the frequency of the constrained system, it could be set to zero. That is to say an asymptotic model $A_{(1)}$ could be used to find the natural frequency of the constrained system $\tilde{A}_{(1)}$. This may be stated mathematically as follows:

$$A_{(1)} \in B_1 \text{ and for } m_1 = 0,\ A_{(1)} = B_1, \tag{9}$$

$$\tilde{A}_{(1)} \in \tilde{B}_1 \text{ and for } m_1 = 0,\ \tilde{A}_{(1)} = \tilde{B}_1. \tag{10}$$

From the existence and convergence theorems which are applicable for systems B_1 and $\tilde{B}_1$,

$$\text{as } k \to \pm\infty, \quad B_1 \to \tilde{B}_1. \tag{11}$$

From Eqs. (9)–(11),

$$\text{as } k \to \pm\infty, \quad A_{(1)} \to \tilde{A}_{(1)}. \tag{12}$$

Although $A_{(1)}$ and $\tilde{A}_{(1)}$ are special cases of B_1 and $\tilde{B}_1$, since $A_{(1)}$ and $\tilde{A}_{(1)}$ are single degree of freedom systems it is not possible to use Eqs. (7) and (8) for these cases with the same notation. To understand the behaviour of the frequency–stiffness relationship in such cases, it is instructive to consider the effect of the mass m_1 on the natural frequencies of B_1:

$$\text{As } m_1 \to 0 \quad \text{for } k>0, \quad \omega_{2,1} \to \infty \text{ and } \omega_{1,1} \to \omega_{1,(1)}, \tag{13}$$

where $\omega_{1,(1)}$ is the first (and only) natural frequency of $A_{(1)}$.

From Eqs. (7a) and (13),

$$\text{as } k \to \infty, \quad \omega_{1,(1)} \to \tilde{\omega}_{1,1}. \tag{14}$$

Here the approach is from below.

It may be noted here that the presence of a mass or inertia along a constrained coordinate does not alter the frequencies. Therefore,

$$\tilde{\omega}_{1,(1)} = \tilde{\omega}_{1,1}. \tag{15}$$

Substituting Eq. (15) into (14) gives

$$k \to \infty, \quad \omega_{1,(1)} \to \tilde{\omega}_{1,(1)}. \tag{16}$$

This is to be expected from Eq. (12) but the above steps show how it occurs.

The behaviour of System $A_{(1)}$ for negative values of stiffness is not so straightforward. The existence of a natural frequency is not guaranteed for all negative values. However, from Eq. (12), it is clear that

$$\text{as } k \to -\infty, \quad \omega_{1,(1)} \to \tilde{\omega}_{1,(1)}. \tag{17}$$

Noting that the natural frequencies of a system cannot increase with a decrease in stiffness, in the above case the natural frequency of the restrained system must approach the natural frequency of the constrained system from above. Eqs. (16) and (17) suggest that the natural frequency $\omega_{1,(1)}$ of $A_{(1)}$ must approach the natural frequency $\tilde{\omega}_{1,(1)}$ of the constrained system $\tilde{A}_{(1)}$ from different sides as the stiffness approaches plus or minus infinity. This is possible only if the frequency squared versus stiffness curve has a pole. The curve on both sides of the pole asymptotically approaches the natural frequency of the constrained system from different sides. The results show that the use of positive and negative stiffness values in asymptotic modelling is justified for this case too. The above example is only an illustration. Using the principle of mathematical induction, it can be shown that the arguments presented here are applicable for systems with any number of constraints, as given in Ref. [4] for the case when the constraints are associated with the motion of a mass or an inertial element. In practice, asymptotic modelling is more likely to be used for determining the natural frequencies of continuous systems. However, for the sake of completeness, the theorems in Ref. [4] need to be restated in a more general way as follows.

3. Existence and convergence theorems for systems with constraints that change the number of degrees of freedom

Theorem (1.A). *If h restraints of positive or negative stiffness are added to an n degree of freedom system* (A) *where* $h<n$, *along coordinates that are directly associated with a mass or an inertial element, then for the resulting system* (A_h), *there exist at least* ($n-h$) *natural frequencies and modes.*

Theorem (1.B). *Furthermore, as the h stiffness parameters approach infinity, the* ($n-h$) *natural frequencies and modes of System* A_h *would asymptotically approach those of the n degree of freedom system subject to h constraints* ($\tilde{A}_h$).

4. Existence and convergence theorems for systems with constraints that do not affect the number of degrees of freedom

Theorem (2.A). *If h restraints of positive or negative stiffness are added to an n degree of freedom system* (A), *along coordinates that are not directly associated with a mass or an inertial element, then for the resulting system* ($A_{(h)}$), *there exist at least* (n) *natural frequencies and modes for all positive values of stiffness, and for very large values of negative stiffness, but for a finite range of negative stiffness values, in the vicinity of h specific critical values some of the frequencies may not exist.*

Theorem (2.B). *As the magnitude of the h stiffness parameters approach infinity, the n natural frequencies and modes of System* A_h *would asymptotically approach those of the n degree of freedom system subject to h constraints* ($\tilde{A}_{(h)}$) *irrespective of the sign of the stiffness.*

5. Concluding remarks

The statements of existence and convergence theorems that justify asymptotic modelling in the determination of natural frequencies have been refined to include cases where the constraints do not change the number of degrees of freedom. For systems that are subject to restraints that are not directly associated with the motion of a mass or an inertial element, some or all of the natural frequencies may not exist in the vicinity of some critical stiffness parameters. In all cases, the natural frequencies of the constrained system are asymptotically approached by the natural frequencies of the system where the constraints are replaced with artificial restraints having very large positive or negative stiffness.

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